

Proof of Andrews' conjecture on a ${}_4\phi_3$ summation

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Abstract. We give a new proof of a ${}_4\phi_3$ summation due to G.E. Andrews and confirm another ${}_4\phi_3$ summation conjectured by him recently. Some variations of these two ${}_4\phi_3$ summations are also given.

Keywords: basic hypergeometric series; Catalan numbers; Andrews' ${}_4\phi_3$ summation

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1 Introduction

Recall that the basic hypergeometric series ${}_{r+1}\phi_r$ [3, p. 4] is defined as

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k z^k}{(q, b_1, b_2, \dots, b_r; q)_k},$$

where $(a_1, \dots, a_m; q)_n = \prod_{i=1}^m ((1 - a_i)(1 - a_i q) \cdots (1 - a_i q^{n-1}))$.

Recently, Andrews [2] gave a new ${}_4\phi_3$ summation formula as follows.

Theorem 1.1 (Andrews). *For $n \geq 0$, there holds*

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq \end{matrix} ; q^2, q^2 \right] = \frac{q^{-n}(a, b, -q; q)_n (ab; q^2)_n}{(ab; q)_n (a, b; q^2)_n}. \quad (1.1)$$

Andrews' identity (1.1) is a deep extension of Shapilo's identity (see [5, p. 123, (5.12)] and [6, p. 31, Ex. 6.C.14])

$$\sum_{k=0}^n C_{2k} C_{2n-2k} = 4^n C_n,$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ are Catalan numbers.

At the end of his paper, Andrews [2] made the following conjecture:

Conjecture 1.2 (Andrews). *For $n \geq 0$, there holds*

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, abq \end{matrix} ; q^2, q^2 \right] \\ &= \frac{(a, -q; q)_n (b; q)_{n-1} (ab; q^2)_{n-1} (abq^{2n-2}(b - q^2) + abq^{n-1}(q - 1) + q - b)}{q^{n+1}(1 - abq^{2n-1})(ab; q)_{n-1}(a, b/q^2; q^2)_n}. \end{aligned} \quad (1.2)$$

He also pointed out that (1.2) is a generalization of the identity [2, (1.11)]

$$\sum_{k=0}^n q^{2k} \mathcal{C}_{2k}(1, -q) \mathcal{C}_{2n-2k+1}(1, -q) = \frac{q^{2n+2}(1 - q^{2n-1})(-q^2; q^2)_{n-1} \mathcal{C}_n(1, -q)}{(-q; q^2)_{n+1}},$$

where $\mathcal{C}_m(\lambda, q) = q^{2m}(-\lambda/q; q^2)_m / (q^2; q^2)_m$ are the q -Catalan numbers introduced in [1].

Andrews proved (1.1) by using the q -binomial theorem and two special cases of the q -Pfaff-Saalschütz summation formula [3, p. 13, (1.7.2)]. In this paper, we first give a new proof of (1.1) along the lines of the proofs in Guo and Zeng [4]. Then we shall prove (1.2) similarly by using (1.1). Some variations of (1.1) and (1.2) are given in the last section.

2 A new proof of Theorem 1.1

For $a = q^2$, the identity (1.1) reduces to

Lemma 2.1. *For $n \geq 0$, there holds*

$$\sum_{k=0}^n \frac{(b, q^{1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} q^{2k} = \frac{q^{-n}(1 - q^{n+1})(1 - bq)(1 - bq^{2n})}{(1 - q)(1 - bq^n)(1 - bq^{n+1})}.$$

Proof. It is easy to verify that

$$\frac{(b, q^{1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} q^{2k} + \frac{(b, q^{1-2n}/b; q^2)_{n-k}}{(q^{2-2n}/b, bq^3; q^2)_{n-k}} q^{2n-2k} = H_k - H_{k+1}, \quad (2.1)$$

where

$$H_k = \frac{q^{k-n}(1 - q^{n-2k+1})(1 - bq)(1 - bq^{2n})(b, bq^{2n-2k+3}; q^2)_k}{(1 - q)(1 - bq^n)(1 - bq^{n+1})(bq, bq^{2n-2k+2}; q^2)_k}.$$

Summing (2.1) over k from 0 to n , we get

$$2 \sum_{k=0}^n \frac{(b, q^{1-2n}/b; q^2)_k}{(q^{2-2n}/b, bq^3; q^2)_k} q^{2k} = H_0 - H_{n+1} = 2H_0,$$

as desired. ■

Proof of Theorem 1.1. Since the ${}_4\phi_3$ series in (1.1) is terminating, it suffices to prove it for $a = q^{2m}$, $m = 1, 2, \dots$. The $a = q^2$ case is true by Lemma 2.1. Let

$$F_k(n, a, b, q) = \frac{(q^{-2n}, a, b, q^{1-2n}/ab; q^2)_k}{(q^2, q^{2-2n}/a, q^{2-2n}/b, abq; q^2)_k} q^{2k},$$

It is not difficult to verify that (or, see [4, (1.1)])

$$F_k(n, a, b, q) - F_k(n, a, b/q^2, q) = \alpha_n F_{k-1}(n-2, aq^2, b, q), \quad (2.2)$$

where

$$\alpha_n = \frac{(b/q^2 - q^{1-2n}/ab)(1-a)(1-aq)(1-q^{-2n})(1-q^{-2n+2})q^2}{(1-ab/q)(1-abq)(1-q^{2-2n}/a)(1-q^{2-2n}/b)(1-q^{4-2n}/b)}.$$

Summing (2.2) over k from 0 to n gives

$$S(n-2, aq^2, b, q) = \alpha_n^{-1} (S(n, a, b, q) - S(n, a, b/q^2, q)), \quad (2.3)$$

where

$$S(n, a, b, q) = \sum_{k=0}^n F_k(n, a, b, q) = \left[\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, abq \end{matrix} ; q^2, q^2 \right].$$

Suppose that (1.1) is true for $a = q^{2m}$. Then by (2.3) we have

$$\begin{aligned} S(n-2, aq^2, b, q) &= \alpha_n^{-1} \left(\frac{q^{-n}(a, b, -q; q)_n (ab; q^2)_n}{(ab; q)_n (a, b; q^2)_n} - \frac{q^{-n}(a, b/q^2, -q; q)_n (ab/q^2; q^2)_n}{(ab/q^2; q)_n (a, b/q^2; q^2)_n} \right) \\ &= \frac{q^{2-n}(aq^2, b, -q; q)_{n-2} (abq^2; q^2)_{n-2}}{(abq^2; q)_{n-2} (aq^2, b; q^2)_{n-2}}. \end{aligned}$$

Replacing n by $n+2$, one sees that (1.1) is true for $aq^2 = q^{2m+2}$. This completes the proof. \blacksquare

3 A proof of Conjecture 1.2

We first consider the $a = q^2$ case of (1.2).

Lemma 3.1. *For $n \geq 0$, there holds*

$$\begin{aligned} &\sum_{k=0}^n \frac{(b, q^{1-2n}/b; q^2)_k}{(q^{4-2n}/b, bq^3; q^2)_k} q^{2k} \\ &= \frac{(1-q^{n+1})(1-bq)(1-bq^{2n-2})(bq^{2n}(b-q^2) + bq^{n+1}(q-1) + q-b)}{q^{n+1}(1-q)(1-b/q^2)(1-bq^{n-1})(1-bq^n)(1-bq^{2n+1})}. \end{aligned} \quad (3.1)$$

Proof. Observe that

$$\frac{(b, q^{1-2n}/b; q^2)_k}{(q^{4-2n}/b, bq^3; q^2)_k} q^{2k} + \frac{(b, q^{1-2n}/b; q^2)_{n-k}}{(q^{4-2n}/b, bq^3; q^2)_{n-k}} q^{2n-2k} = H_k - H_{k+1}, \quad (3.2)$$

where

$$\begin{aligned} H_k &= \frac{(1-q^{n-2k+1})(1-bq)(1-bq^{2n-2})(b^2q^{2n-1} - bq^{2n-2k+1} + bq^n(q-1) - bq^{2k-1} + 1)}{q^{n-k}(1-q)(1-b/q^2)(1-bq^{n-1})(1-bq^n)(1-bq^{2n+1})} \\ &\quad \times \frac{(b/q^2, bq^{2n-2k+3}; q^2)_k}{(bq, bq^{2n-2k}; q^2)_k}. \end{aligned}$$

Then summing (3.2) over k from 0 to n , we obtain (3.1). ■

Noticing that

$$\frac{(q^{3-2n}/ab; q^2)_k}{(q^{4-2n}/b; q^2)_k} = \frac{(1 - 1/aq)}{(1 - q^{1-2n}/ab)} \frac{(q^{1-2n}/ab; q^2)_k}{(q^{4-2n}/b; q^2)_k} + \frac{(1/aq - q^{1-2n}/ab)}{(1 - q^{1-2n}/ab)} \frac{(q^{1-2n}/ab; q^2)_k}{(q^{2-2n}/b; q^2)_k},$$

we have

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, abq \end{matrix} ; q^2, q^2 \right] \\ &= \frac{bq^{2n-2}(1-aq)}{(1-abq^{2n-1})} \left[\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, abq \end{matrix} ; q^2, q^2 \right] \\ &+ \frac{(1-bq^{2n-2})}{(1-abq^{2n-1})} \left[\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq \end{matrix} ; q^2, q^2 \right]. \end{aligned} \quad (3.3)$$

By (3.3) and (1.1), one sees that (1.2) is equivalent to the following result.

Theorem 3.2. *For $n \geq 0$, there holds*

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{4-2n}/b, abq \end{matrix} ; q^2, q^2 \right] \\ &= \frac{(a, -q; q)_n (b; q)_{n-1} (ab; q^2)_{n-1} (abq^{2n-2}(b-q^2) + abq^{n-1}(q-1) + q-b)}{bq^{3n-1}(1-aq)(ab; q)_{n-1}(a, b/q^2; q^2)_n} \\ &- \frac{(a, b, -q; q)_n (ab; q^2)_n}{bq^{3n-2}(1-aq)(ab; q)_n (a, q^2)_n (b; q^2)_{n-1}}. \end{aligned} \quad (3.4)$$

Proof. Let

$$F_k(n, a, b, q) = \frac{(q^{-2n}, a, b, q^{1-2n}/ab; q^2)_k}{(q^2, q^{2-2n}/a, q^{4-2n}/b, abq; q^2)_k} q^{2k},$$

Similarly to (3.6), we have

$$F_k(n, a, b, q) - F_k(n, a/q^2, b, q) = \beta_n F_{k-1}(n-2, a, bq^2, q), \quad (3.5)$$

where

$$\beta_n = \frac{(a/q^2 - q^{1-2n}/ab)(1-b)(1-bq)(1-q^{-2n})(1-q^{-2n+2})q^2}{(1-ab/q)(1-abq)(1-q^{2-2n}/a)(1-q^{4-2n}/a)(1-q^{4-2n}/b)}.$$

Summing (3.5) over k from 0 to n yields that

$$S(n, a, b, q) - S(n, a/q^2, b, q) = \beta_n S(n-2, a, bq^2, q), \quad (3.6)$$

where $S(n, a, b, q)$ denotes the left-hand side of (3.4).

It suffices to prove (3.4) for $a = q^{2m}$, $m = 1, 2, \dots$. The $a = q^2$ case is true by (3.3), (1.1) and Lemma 3.1. We then can complete the proof of (3.4) by induction on n (firstly)

and m (secondly) by checking that the right-hand side of (3.4) also satisfies the relation (3.6). \blacksquare

Remark. One may wonder, why not prove (1.2) directly by induction? The reason is that we cannot find a simple recurrence relation like (3.6) for the ${}_4\phi_3$ series in (1.2). It is also worth mentioning that the $a = q^2$ case of (3.4) cannot be proved in the same way as (3.1). This makes our proof of Conjecture 1.2 a bit complicated and not so straightforward.

4 Concluding remarks

Letting $(a, b, q) \rightarrow (a^{-1}, b^{-1}, q^{-1})$ in (1.1), we obtain the following variation

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq \end{matrix} ; q^2, q^4 \right] = \frac{(a, b, -q; q)_n (ab; q^2)_n}{(ab; q)_n (a, b; q^2)_n}. \quad (4.1)$$

Since

$$(q^{3-2n}/ab; q^2)_k = \frac{1}{1 - q^{1-2n}/ab} (q^{1-2n}/ab; q^2)_k - \frac{q^{1-2n+2k}/ab}{1 - q^{1-2n}/ab} (q^{1-2n}/ab; q^2)_k,$$

combining (1.1) and (4.1) leads to

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{3-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq \end{matrix} ; q^2, q^2 \right] = \frac{(a, b, -q; q)_n (ab; q^2)_n}{(1 - abq^{2n-1})(ab; q)_{n-1} (a, b; q^2)_n}.$$

Moreover, replacing b by bq^2 in (1.2), we have

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, bq^2, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq^3 \end{matrix} ; q^2, q^2 \right] \\ &= \frac{(a, -q; q)_n (bq^2; q)_{n-1} (abq^2; q^2)_{n-1} (abq^{2n+1}(b-1) + abq^n(q-1) + 1 - bq)}{q^n(1 - abq^{2n+1})(abq^2; q)_{n-1} (a, b; q^2)_n}. \end{aligned} \quad (4.2)$$

Substituting $(a, b, q) \rightarrow (a^{-1}, b^{-1}, q^{-1})$ in (4.2), we get

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, bq^2, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq^3 \end{matrix} ; q^2, q^4 \right] \\ &= \frac{(a, -q; q)_n (bq^2; q)_{n-1} (abq^2; q^2)_{n-1} (abq^{2n}(bq-1) + bq^n(1-q) + 1 - b)}{(1 - abq^{2n+1})(abq^2; q)_{n-1} (a, b; q^2)_n}. \end{aligned} \quad (4.3)$$

Since

$$(aq^2; q^2)_k = \frac{(a; q^2)_k}{1 - a} - \frac{a(a; q^2)_k q^{2k}}{1 - a},$$

combining (4.2) and (4.3) immediately yields that

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, aq^2, bq^2, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq^3 \end{matrix} ; q^2, q^2 \right] = \frac{q^{-n}(aq, bq, -q; q)_n (abq^2; q^2)_n}{(1 - abq^{2n+1})(abq^2; q)_{n-1} (a, b; q^2)_n}. \quad (4.4)$$

Noticing that

$$\frac{(q^{-2n}; q^2)_k}{(q^2; q^2)_k} = \frac{(q^{-2n-2}; q^2)_k}{(q^2; q^2)_k} + q^{-2n-2} \frac{(q^{-2n}; q^2)_{k-1}}{(q^2; q^2)_{k-1}}$$

and $(x; q)_k = (1 - x)(xq; q)_{k-1}$, we have

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{-1-2n}/ab \\ q^{-2n}/a, q^{-2n}/b, abq \end{matrix} ; q^2, q^2 \right] \\ &= {}_4\phi_3 \left[\begin{matrix} q^{-2n-2}, a, b, q^{-1-2n}/ab \\ q^{-2n}/a, q^{-2n}/b, abq \end{matrix} ; q^2, q^2 \right] \\ &+ \frac{q^{-2n}(1-a)(1-b)(1-q^{-1-2n}/ab)}{(1-q^{-2n}/a)(1-q^{-2n}/b)(1-abq)} {}_4\phi_3 \left[\begin{matrix} q^{-2n}, aq^2, bq^2, q^{1-2n}/ab \\ q^{2-2n}/a, q^{2-2n}/b, abq^3 \end{matrix} ; q^2, q^2 \right]. \quad (4.5) \end{aligned}$$

Plugging the formulas (1.1) ($n \rightarrow n+1$) and (4.4) into (4.5), and making some simplification, we obtain the following new neat ${}_4\phi_3$ summation formula:

$${}_4\phi_3 \left[\begin{matrix} q^{-2n}, a, b, q^{-1-2n}/ab \\ q^{-2n}/a, q^{-2n}/b, abq \end{matrix} ; q^2, q^2 \right] = \frac{(aq, bq, -q; q)_n (abq^2; q^2)_n}{(abq; q)_n (aq^2, bq^2; q^2)_n}.$$

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